

The random field Ising model with an asymmetric trimodal probability distribution

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Abstract

The Ising model in the presence of a random field is investigated within the mean field approximation based on Landau expansion. The random field is drawn from the trimodal probability distribution $P(h_i) = p\delta(h_i - h_0) + q\delta(h_i + h_0) + r\delta(h_i)$, where the probabilities p, q, r take on values within the interval $[0, 1]$ consistent with the constraint $p + q + r = 1$ (asymmetric distribution), h_i is the random field variable and h_0 the respective strength. This probability distribution is an extension of the bimodal one allowing for the existence in the lattice of non magnetic particles or vacant sites. The current random field Ising system displays second order phase transitions, which, for some values of p, q and h_0 , are followed by first order phase transitions, thus confirming the existence of a tricritical point and in some cases two tricritical points. Also, reentrance can be seen for appropriate ranges of the aforementioned variables. Using the variational principle, we determine the equilibrium equation for magnetization, solve it for both transitions and at the tricritical point in order to determine the magnetization profile with respect to h_0 .

Key words: Ising model, mean-field approximation, trimodal random field, Landau theory, phase-diagram, tricritical point, phase transitions

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1 Introduction

The pure models for crystalline materials can describe the respective experimental samples in exceptional situations, since such a sample can contain impurities, broken bonds, defects, etc., making real physical systems to never be translationally invariant, thus necessitating the modification of pure models appropriately for comparing the experimental results with the theoretical predictions. A small amount of quenched randomness can influence significantly the phase transitions replacing a first-order phase transition (FOPT) by a second-order phase transition (SOPT), so that tricritical points and critical-end-points are suppressed [1]. In two dimensions, an infinitesimal amount of field randomness destroys any FOPT. One such situation is the presence of random magnetic fields acting on each spin in an otherwise free of defects lattice; the respective pure system is considered to be described by an Ising model, which is now transformed into the random field Ising model (RFIM) in the presence of random fields [2,3,4]. RFIM had been the standard vehicle for studying the effects of quenched randomness on phase diagrams and critical properties of lattice spin-systems and had been studied for many years since the seminal work of Imry and Ma [4]. Associated with this model are the notions of lower critical dimension, tricritical points, higher order critical points and random-field probability distribution function (PDF). The simplest model exhibiting a tricritical phase diagram in the absence of randomness is the Blume-Capel model – a regular Ising spin-1 model [5,6]. Although much effort has been invested for the study of the RFIM, the only well-established conclusion is the existence of a phase transition for $d \geq 3$ (d space dimension), that is, the critical lower dimension d_l is 2 after a long controversial discussion [4,7], while many other issues are still unanswered; among them is the order of the phase transition, the existence of a tricritical point (TCP) and the dependence of these on the form of the random field PDF. According to the mean-field approximation (MFA) the choice of the random field PDF can lead to a continuous ferromagnetic/paramagnetic (FM/PM) boundary as in the single Gaussian, whereas for the bimodal this boundary can be divided into two parts, an SOPT branch for high temperatures and an FOPT branch for low temperatures separated by a TCP at $kT_c^t/(zJ) = 2/3$ and $h_c^t/(zJ) = (kT_c^t/(zJ)) \times \arg \tanh(1/\sqrt{3}) \simeq 0.439$ [8,9,10], where z is the coordination number, k is the Boltzmann constant and T_c^t, h_c^t are the tricritical temperature and random field, respectively, such that for $T < T_c^t$ and $h > h_c^t$ the transition to the FM phase is of first order. However, this behavior is not fully elucidated since in the case of the three dimensional RFIM, the high temperature series expansions yield only continuous transitions for both PDF's [11]; according to Houghton et al [12] both distributions predict a tricritical point with $h_c^t = 0.28 \pm 0.01$ and $T_c^t = 0.49 \pm 0.03$ for the bimodal and $\sigma_c^t = 0.36 \pm 0.01$ and $T_c^t = 0.36 \pm 0.04$ for the Gaussian with critical standard deviation σ_c^t . Galam and Birman studied the crucial issue for the

existence of a TCP within the mean-field theory for a general PDF $p(\vec{H})$ by using an even-degree free energy expansion up to eighth degree in the order parameter; they proposed some inequalities between the derivatives of the PDF up to sixth order at zero magnetic field for the possible existence of a TCP [13]. In Monte Carlo studies for $d = 3$, Machta et al [14], using the Gaussian distribution, could not reach a definite conclusion concerning the nature of the transition, since for some realizations of randomness the magnetization histogram was two-peaked (implying an SOPT) whereas for other ones was three-peaked implying an FOPT; Middleton and Fisher [15], using a similar distribution for $T = 0$, suggested an SOPT with a small order-parameter exponent $\beta = 0.017(5)$; Fytas et al [16], following Wang-Landau and Lee entropic sampling schemes for the bimodal distribution function with $h_0 = 2$ and $h_0 = 2.25$ for a simple cubic lattice, concluded that their results indicated an SOPT by applying the Lee-Kosterlitz free energy barrier method; Hernández and coworkers claim they have found a crossover between an SOPT and an FOPT at a finite temperature and magnetic field for the bimodal distribution function [17]. One of the main issues was the experimental realization of random fields. Fishman and Aharony [18] showed that the randomly quenched exchange interactions Ising antiferromagnet in a uniform field H is equivalent to a ferromagnet in a random field with the strength of the random field linearly proportional to the induced magnetization. Also another interesting result found by Galam [19] via MFA was that the Ising antiferromagnets in a uniform field with either a general random site exchange or site dilution have the same multicritical space as the random-field Ising model with bimodal PDF.

The usual PDF for the random field is either the symmetric bimodal

$$P(h_i) = p\delta(h_i - h_0) + q\delta(h_i + h_0) \quad (1)$$

where p is the fraction of lattice sites having a magnetic field h_0 , while the rest fraction has a field $(-h_0)$ and $p = q = \frac{1}{2}$ [8,20,21], or the Gaussian, single or double symmetric,

$$P(h_i) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left[-\frac{h_i^2}{2\sigma^2}\right] \\ P(h_i) = \frac{1}{2} \frac{1}{(2\pi\sigma^2)^{1/2}} \left\{ \exp\left[-\frac{(h_i - h_0)^2}{2\sigma^2}\right] + \exp\left[-\frac{(h_i + h_0)^2}{2\sigma^2}\right] \right\} \quad (2)$$

with mean value zero and $(h_0, -h_0)$, respectively, and standard deviation σ [9,22].

Galam and Aharony, in a series of investigations, presented a detailed analysis

via mean field and renormalization group of a system consisting of n -component classical spins (finally choosing $n = 3$) on a d -dimensional lattice of a uniaxially anisotropic ferromagnet in a longitudinal random field extracted from a symmetric bimodal PDF ($p = q = 1/2$) without and with a uniform magnetic field along the easy axis, respectively [23,24]. The uniaxial anisotropy was chosen to be along the easy axis and the exchange couplings were of the form $J^{(2)} = aJ^{(1)}$, where a is the anisotropy and $0 \leq a \leq 1$. Depending on the anisotropy (small, medium, large) a variety of phases (longitudinal, transverse, paramagnetic), critical, bicritical, critical-end points as well as a multicritical point (an intersection of bicritical, tricritical and critical-end-point lines) resulted. In addition to these purely theoretical investigations, Galam, proposed a model (diluted random field) in his attempt to reproduce some of the features in the phase diagram of the experimental sample consisting of the mixed cyanide crystals $X(CN)_xY_{1-x}$, where X stands for an alkali metal (K,Na,Rb) and Y a spherical halogen ion (Br,Cl,I); the dilution of the pure crystal XCN is achieved by replacing CN by the halogen ions Y [25]. The pure alkali-cyanide XCN crystal ferroelastic transition disappears at some concentration x_c of the cyanide; its numerical value depends on both components X, Y . By choosing a model Hamiltonian (ferromagnetic Ising-type with nearest neighbor interaction) with dilution and a symmetric trimodal PDF for the random fields, Galam, using MFA, managed to predict the involved first and second order phase transitions with the interfering TCP as well as the respective concentration for a phase transition to occur depending on the procedure considered. The random fields were necessary because there were experimental evidences that below x_c cyanide displayed orientational freezing and the random fields were used for fixing this orientation. The involved probability p_t in PDF as well as the critical threshold x_c were expressed in terms of microscopic quantities.

Recently, the asymmetric bimodal PDF (1) with $p \neq q$, in general, has also been studied in detail [26]. This study has revealed that for some values of p and h_0 the PM/FM boundary is exclusively of second order; however, for some other ranges of these variables this boundary consists of two branches, a second order one and another of first order, thus confirming the existence of a tricritical point, whose temperature depends only on the probability p in (1). In addition to these findings, the occurrence of reentrance has been corroborated as well as complex magnetization profiles with respect the random field strength h_0 . For $p = q$, symmetric bimodal PDF, the results found by Aharony were confirmed [8].

An immediate generalization of the asymmetric bimodal (1) is the asymmetric trimodal one,

$$P(h_i) = p\delta(h_i - h_0) + q\delta(h_i + h_0) + r\delta(h_i) \quad (3)$$

where $p+q+r = 1$. In earlier studies, the probabilities p, q had been considered as equal and related to r by the relation $p = q = (1 - r)/2$, symmetric PDF [27,28]. The third-peak, introduced in addition the other two ones in the bimodal (1) and associated with the third term in (3), is to allow for the presence of non magnetic spins or vacancies in the lattice that are not affected by the random magnetic fields and reduces the randomness of the system, as well.

For the critical exponents of the three-dimensional RFIM, it seems that there is broad consensus concerning their values except for the specific heat exponent α , for which there is much dispute concerning its numerical value, since its sign is widely accepted to be negative. The main source of information for the critical exponents are Monte Carlo simulations. However, they provide various values depending on the probability distribution considered. Middleton and Fisher concluded that the α -exponent is near zero, $\alpha = -0.01 \pm 0.09$ [15]. Rieger and Young, considering the bimodal distribution, found $\alpha = -1.0 \pm 0.3$ [29], Rieger, using the single Gaussian distribution, found $\alpha = -0.5 \pm 0.2$ [30], whereas Hartmann and Young, in ground-state calculations, found $\alpha = -0.63 \pm 0.07$ [31]. Nowak et al concluded that $\alpha = -0.5 \pm 0.2$ [32], whereas Dukovski and Machta found a positive value, namely, $\alpha = 0.12$ [33]. Malakis and Fytas [34], by applying the critical minimum-energy subspace scheme in conjunction with the Wang-Landau and broad-histogram methods for cubic lattices, proved that the specific heat and susceptibility are non-self-averaging for $d = 3$ using the bimodal distribution. The same ambiguous situation prevails in experimental measurements, see Ref. [35].

In this work, we study the RFIM with the asymmetric trimodal PDF (3) with arbitrary values for the probabilities p, q ($p+q+r = 1$) in order to investigate the phase diagrams, phase transitions, tricritical points and magnetization profiles with respect to h_0 . The paper is organized as follows: In the next section, the suitable Hamiltonian is introduced and the respective free energy and equation of state for the magnetization are derived. In section 3, the phase diagram, tricritical points and magnetization profiles for various values of p and q are calculated and discussed; we close with the conclusions in section 4.

2 The model

The pure Ising model Hamiltonian in the presence of random fields changes into,

$$H = -J \sum_{\langle i,j \rangle} S_i S_j - \sum_i h_i S_i, \quad S_i = \pm 1 \quad (4)$$

The summation in the first term extends over all nearest neighbors and is denoted by $\langle i, j \rangle$; in the second term h_i represents the random field that

couples to the one-dimensional spin variable S_i . We also consider that $J > 0$ so that the ground state is ferromagnetic in the absence of random fields. The presence of randomness necessitates considering two averaging procedures, the usual thermal average, denoted by angular brackets $\langle \dots \rangle$, and the disorder average over the random fields denoted by $\langle \dots \rangle_h$ for the respective PDF. We also make assumptions concerning the random field h_i ,

$$\langle h_i \rangle_h = (p - q)h_0, \quad \langle h_i h_j \rangle_h = h_0^2 \delta_{ij} \quad (5)$$

The former relation in (5) vanishes for a symmetric PDF ($p = q$), whereas for the asymmetric PDF ($p \neq q$) it is non zero implying that the system has residual magnetization, thereby affecting considerably the system's magnetization; a similar case has appeared in Ref. [26]. The latter one implies that there is no correlation between h_i at different lattice sites.

According to the MFA, the Hamiltonian (4) takes the form [8,9,20,26],

$$H_{MFA} = \frac{1}{2} N z J M^2 - \sum_i (z J M + h_i) S_i \quad (6)$$

where N is the number of spins in the lattice and M the magnetization; the respective free energy per spin within the MFA is,

$$\begin{aligned} \frac{1}{N} \langle F \rangle_h &= \frac{1}{2} z J M^2 - \frac{1}{\beta} \langle \ln \{ 2 \cosh [\beta (z J M + h_i)] \} \rangle_h \\ &= \frac{1}{2} z J M^2 - \frac{1}{\beta} \int P(h_i) \ln \{ 2 \cosh [\beta (z J M + h_i)] \} dh_i \end{aligned} \quad (7)$$

where the probability $P(h_i)$ is chosen to be the trimodal (3) and p, q take on any value within the interval $[0,1]$, consistent with the relation $p + q + r = 1$, and $\beta = 1/(kT)$.

The magnetization is the solution to the equation $d(\langle F \rangle_h / N) / dM = 0$, equilibrium condition,

$$M = \langle \tanh [\beta (z J M + h_i)] \rangle_h \quad (8)$$

If the distribution $P(h_i)$ is symmetric, $P(h_i) = P(-h_i)$, which occurs for $p = q = (1 - r)/2$, then the value $M = 0$ (PM phase) will always be a solution to (8), otherwise this is not if $P(h_i)$ is non symmetric, $p \neq q$; however, this can be remedied if an auxiliary field V_0 is introduced into the system such that [8,26],

$$\langle \tanh [\beta (h_i + V_0)] \rangle_h = 0 \quad (9)$$

inducing, in this way, the PM phase. However, this relation acts also as a constraint on the system under consideration influencing, nevertheless, its behavior. The free energy (7), in the presence of the auxiliary field V_0 , takes now the form,

$$\begin{aligned}\frac{1}{N}\langle F \rangle_h &= \frac{1}{2}zJM^2 - \frac{1}{\beta}\langle \ln\{2 \cosh[\beta(zJM + h_i + V_0)]\} \rangle_h \\ &= \frac{1}{2}zJM^2 - \frac{1}{\beta}\left\{F_0 + \frac{\alpha^2 F_2}{2!}M^2 + \frac{\alpha^3 F_3}{3!}M^3 + \frac{\alpha^4 F_4}{4!}M^4 + \right. \\ &\quad \left. \frac{\alpha^6 F_6}{6!}M^6\right\}\end{aligned}\quad (10)$$

after expanding the quantity in angular brackets in powers of M and calculating the average values using (3) with $\alpha \equiv \beta Jz$. By setting $t_i \equiv \tanh[\beta(V_0 + h_i)]$, $t_+ \equiv \tanh[\beta(V_0 + h_0)]$, $t_- \equiv \tanh[\beta(V_0 - h_0)]$ and $t_0 \equiv \tanh[\beta V_0]$, we have,

$$\begin{aligned}F_0 &= \langle \ln\{2 \cosh[\beta(V_0 + h_i)]\} \rangle_h \\ &= \ln 2 + p \ln \cosh[\beta(V_0 + h_0)] + q \ln \cosh[\beta(V_0 - h_0)] + r \ln \cosh[\beta V_0] \\ F_1 &= \langle t_i \rangle_h = pt_+ + qt_- + rt_0 \\ F_2 &= \langle 1 - t_i^2 \rangle_h = 1 - pt_+^2 - qt_-^2 - rt_0^2 \\ F_3 &= \langle -2t_i(1 - t_i^2) \rangle_h \\ &= -2pt_+(1 - t_+^2) - 2qt_-(1 - t_-^2) - 2rt_0(1 - t_0^2) \\ F_4 &= \langle 2(1 - t_i^2)(3t_i^2 - 1) \rangle_h \\ &= 2p(1 - t_+^2)(3t_+^2 - 1) + 2q(1 - t_-^2)(3t_-^2 - 1) + 2r(1 - t_0^2)(3t_0^2 - 1) \\ F_6 &= \langle 8(1 - t_i^2)(15t_i^4 - 15t_i^2 + 2) \rangle_h \\ &= 8p(1 - t_+^2)(15t_+^4 - 15t_+^2 + 2) + 8q(1 - t_-^2)(15t_-^4 - 15t_-^2 + 2) \\ &\quad + 8r(1 - t_0^2)(15t_0^4 - 15t_0^2 + 2)\end{aligned}\quad (11)$$

The condition (9) for the existence of the PM phase is equivalent to setting $F_1 = 0$,

$$pt_+ + qt_- + rt_0 = 0 \quad (12)$$

The equilibrium condition $d(\langle F \rangle_h/N)/dM = 0$ yields,

$$M = \alpha F_2 M + \frac{\alpha^2 F_3}{2!}M^2 + \frac{\alpha^3 F_4}{3!}M^3 + \frac{\alpha^5 F_6}{5!}M^5 \quad (13)$$

or,

$$M = AM + BM^2 + CM^3 + EM^5 \quad (14)$$

$$A \equiv \alpha F_2, B \equiv \frac{\alpha^2 F_3}{2!}, C \equiv \frac{\alpha^3 F_4}{3!}, E \equiv \frac{\alpha^5 F_6}{5!} \quad (15)$$

In RFIM if there is a phase transition it will be associated with the magnetization and the possible two phases are the PM with $M = 0$ and FM with $M \neq 0$. The phase boundary is found by solving Eq. (14) in conjunction with the free energy (10) in case of an FOPT. The SOPT boundary is determined by setting $A = 1$ and $C < 0$, whereas the FOPT boundary by $A = 1$ and $C > 0$. These two boundaries, whenever appear sequentially, are joined at a tricritical point determined by the condition $A = 1$ and $C = 0$ [8,20,21,22,26,36,37,38], provided that $E < 0$ (equivalently, $F_6 < 0$) for stability, as also in [36,37]. However, for the FOPT boundary we shall use, in addition to (12) and (14), the requirement of the equality of the respective free energies, $F(M = 0) = F(M \neq 0)$, where $F \equiv \langle F \rangle_h / N$.

3 Phase diagram. Magnetization profiles

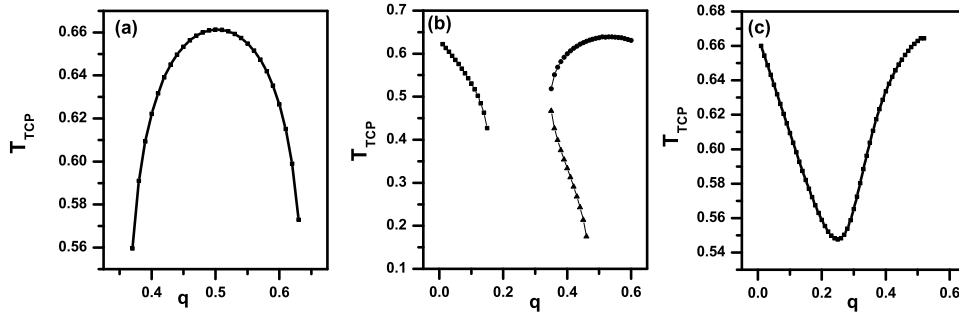


Fig. 1. The tricritical temperature against the probability q for specific values of p . In panel (a) ($p = 0.01$) it varies non-monotonically with a maximum value at $q = 0.50$. In panel (b) ($p = 0.40$) the left-hand-side points correspond to tricritical temperatures for low q -values ($0.0 \leq q \leq 0.15$). The right-hand-side group of points forms two branches, the upper one refers to the upper tricritical temperatures ($0.35 \leq q \leq 0.60$), whereas the lower one corresponds to the lower tricritical temperatures for fewer values of q , $0.35 \leq q \leq 0.46$. In panel (c) ($p = 0.48$) the tricritical temperature varies non-monotonically with a minimum at $q = 0.25$.

Using the conditions for the calculation of the TCP together with Eq. (12), the TCP coordinates $(T^{TCP}, h_0^{TCP}, V_0^{TCP})$ are calculated as functions of the probabilities p, q . These exist only for a limited number of p 's and q 's, namely, $p \in [0.0, 0.63]$ whereas the respective q -values depend on the specific p -values, but they lie in same interval, as well. However, the tricritical temperature T^{TCP} does not satisfy any simple closed-formula as in the asymmetric bimodal PDF [26]. The resulting tricritical temperatures exhibit a variety of variations as functions of p, q , see Fig. 1. However, for some p - and q -values two tricritical temperatures (upper and lower ones) occur, see Fig. 1(b) [39,40], since both temperatures are solutions to the simultaneous equations $\alpha F_2 = 1, F_4 = 0$.

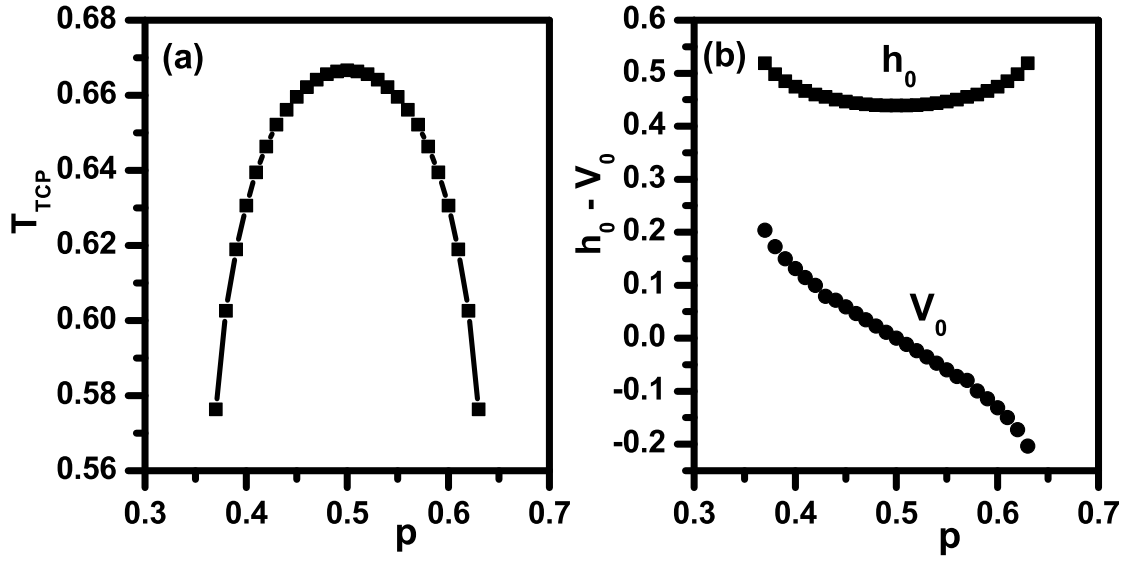


Fig. 2. Tricritical point coordinates for the bimodal PDF, (a) tricritical temperature and (b) random field h_0 and auxiliary field V_0 , resulting form trimodal PDF in the limiting case $r = 0$. The agreement of these figures with the corresponding ones in Ref. [26] is complete.

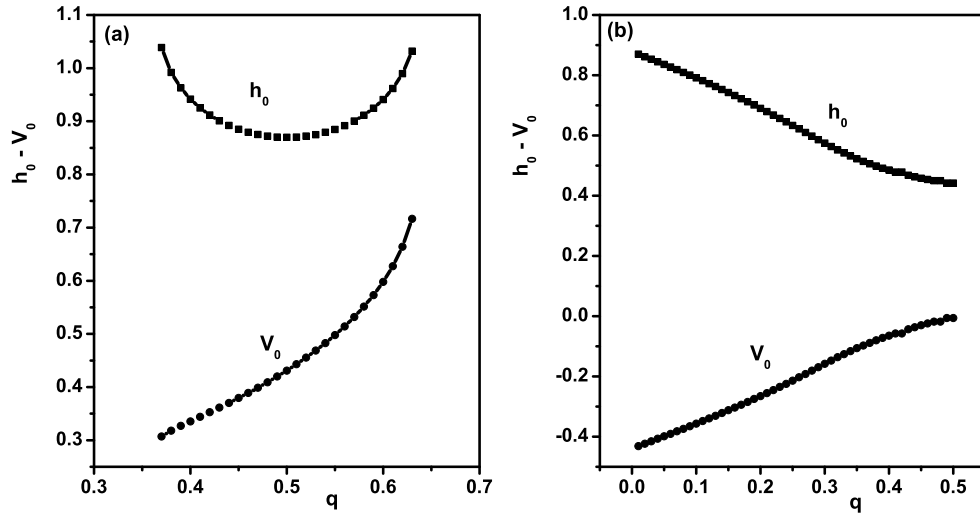


Fig. 3. Variation of the random field strength h_0 and auxiliary field V_0 with q for specific p -values at the tricritical point for the trimodal PDF. The random field h_0 exhibits a non-monotonic behavior in (a) for $p = 0.01$ and monotonic in (b) for $p = 0.50$. The auxiliary potential V_0 increases monotonically with q for any p . Both quantities are in units of (Jz) , i.e., $h_0 \equiv h_0/(Jz)$, $V_0 \equiv V_0/(Jz)$.

In order to examine the validity of the process under consideration, we focus on the asymmetric bimodal PDF resulting from the trimodal by setting $r = 0$ and studied earlier [26]; using the data for $r = 0$, we recover exactly the same plots for the tricritical temperature as well as h_0, V_0 , see Fig. 2.

Now, using the relations $A = 1$ and $C = 0$ (equivalently, $\alpha F_2 = 1$ and $F_4 = 0$) together with (12), the remaining coordinates h_0^{TCP} and V_0^{TCP} for the TCP are calculated, see Fig. 3. Both quantities display two modes of variation; in mode-1 ($p \leq 0.49$) h_0^{TCP} varies non-monotonically with q , Fig. 3(a); in mode-2 ($p \geq 0.50$) h_0^{TCP} decreases monotonically with q , Fig. 3(b). However, in both modes, V_0^{TCP} increases monotonically, but for type-1 the increase is more steep than in type-2.

The magnetization at the TCP is found by solving Eq. (13) taking into consideration the appropriate conditions for the TCP,

$$\frac{\alpha^2 F_3}{2!} M^2 + \frac{\alpha^5 F_6}{5!} M^5 = 0 \quad (16)$$

or

$$F_6 \omega^5 + 60 F_3 \omega^2 = 0 \quad (17)$$

where $\omega \equiv \alpha M$, whose solutions are,

$$\omega_1^{TCP} = 0 \quad (18)$$

$$\omega_2^{TCP} = \left(-60 F_3 / F_6 \right)^{1/3} \quad (19)$$

from which the magnetizations $M_{1,2}^{TCP} = \omega_{1,2}^{TCP} * (kT^{TCP} / (Jz))$ can be deduced. The first one, (18), is the magnetization of the PM phase ($M_1^{TCP} = 0$), whereas the second one, (19), is the magnetization of the FM phase ($M_2^{TCP} \neq 0$). However, the nonzero solution M_2^{TCP} , in general, has lower free energy than the zero solution (18), implying that this is the stable solution at the tricritical point, see Fig. 4. According to the first relation in Eqs. (5), for the general case $p \neq q$, the mean value of the random field is non zero; this is equivalent to the presence of an external magnetic field in the system so that the magnetization at the tricritical point scales as $M_t \equiv M(T = T^{TCP}) \sim h_{TCP}^{1/\delta_t}$, where h_{TCP} is the magnetic field and the tricritical exponent $\delta_t = 5$ according to the Landau theory [41,42,43]. In case of equal partial probabilities ($p = q$), the M_2^{TCP} -magnetization vanishes ($M_2^{TCP} = 0 = M_1^{TCP}$) and the system has only the zero solution (double root), which now becomes the stable one. Representative plots of the magnetization M_2^{TCP} with respect to q appear in Fig. 5 for specific values of p .

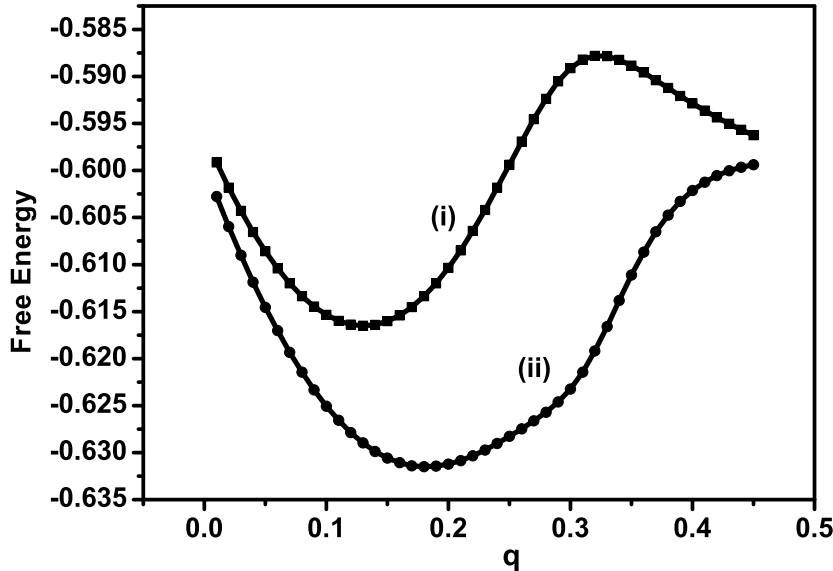


Fig. 4. Free energy of the solutions (18) and (19) at the tricritical point for $p = 0.55$; the upper plot (i) corresponds to the zero solution M_1 and the lower (ii) to the nonzero solution M_2 . For this p -value, the M_2 is the stable solution, whereas the M_1 is metastable; this happens, in general, for other p -values. The M_2 solution coincides with the zero one (M_1) only for $p = q$.

In a previous communication [26], the PDF of the RFIM was selected to be the asymmetric bimodal (1); this system displayed a symmetric behavior at the tricritical point with respect to the probability p ; especially, two distinct tricritical points with probabilities p_1 and p_2 , respectively, such that $p_1 + p_2 = 1$, have identical tricritical temperatures and random fields, whereas the respective auxiliary fields and non zero magnetizations are absolutely equal. A similar behavior is also displayed by the present model with respect to the probabilities p and q ; if the probabilities (p_1, q_1) and (p_2, q_2) of the tricritical points of two systems are interchanged, namely, $p_2 = q_1, q_2 = p_1$, then these systems have the same tricritical temperatures and random fields, whereas the respective auxiliary fields V_0^{TCP} and the nonzero magnetizations M_2^{TCP} are absolutely equal. These systems also have equal the respective free energies for the zero magnetization ($F(p_1, q_1, M_1^{TCP} = 0) = F(p_2, q_2, M_1^{TCP} = 0)$) as well as the nonzero one ($F(p_1, q_1, M_2^{TCP}) = F(p_2, q_2, M_2^{TCP})$); the latter result implies that the two magnetizations $M_2^{TCP}(p_1, q_1), M_2^{TCP}(p_2, q_2)$ are equally probable, an expected result, since the magnetizations have equal absolute values, the only difference being in their sign, implying that no direction is favored.

After the calculation of the tricritical point coordinates $(T^{TCP}, h_0^{TCP}, V_0^{TCP})$ for the respective values of p and q , we proceed to determine the phase diagram

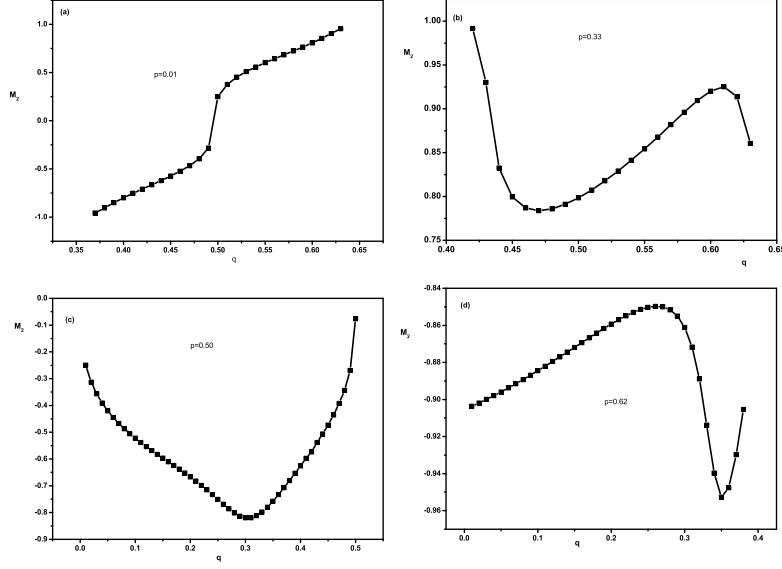


Fig. 5. Variation of the nonzero magnetization $M_2^{TCP} = (-60F_3/F_6)^{1/3}$ with q for specific p -values at the tricritical point. M_2 is, in general, the stable solution. Except for $p = 0.01$, in the other panels the TCP-magnetization displays a non monotonic behavior as a function of q .

by solving Eq. (14), taking into consideration the respective conditions for the FOPT and the SOPT. By varying the parameters p and q many different types of phase diagrams result, as those appearing in Fig. 6(a) labelled by the individual p and q values. The curves in Fig. 6(a) are classified into two main groups: the first one includes those curves that do not possess a TCP corresponding to an SOPT only, curves (i) and (ii); the second group includes those having a TCP, which joins the FOPT-branch with the SOPT-branch of the phase diagram, curves (iii) and (iv). The occurrence of an FOPT, and subsequently of a TCP, results from the competition between the first term in the Hamiltonian (4) (tending to make parallel the spins) and the second term of random forces inducing disorder. For small values of h_0 the competition is weak allowing the first term to dominate, but as h_0 increases the second term dominates over the first one, thus changing the phase transition from second order to first order. Some of the curves in the latter group have a second tricritical point and/or present reentrance Fig. 6(a(iv)). Reentrance might also be attributed to the competition between these two terms in the Hamiltonian (4). In the phenomenon of reentrance a vertical line in the (h_0, T) -plane crosses the transition line twice, in that, by lowering the temperature at constant h_0 , one observes first a PM/FM transition and then, on further lowering the temperature, an FM/PM transition appears so that the magnetization is zero although the temperature is low and the system remains in the PM phase for these temperatures, Fig. 6(b(i)) or another transition may take place from

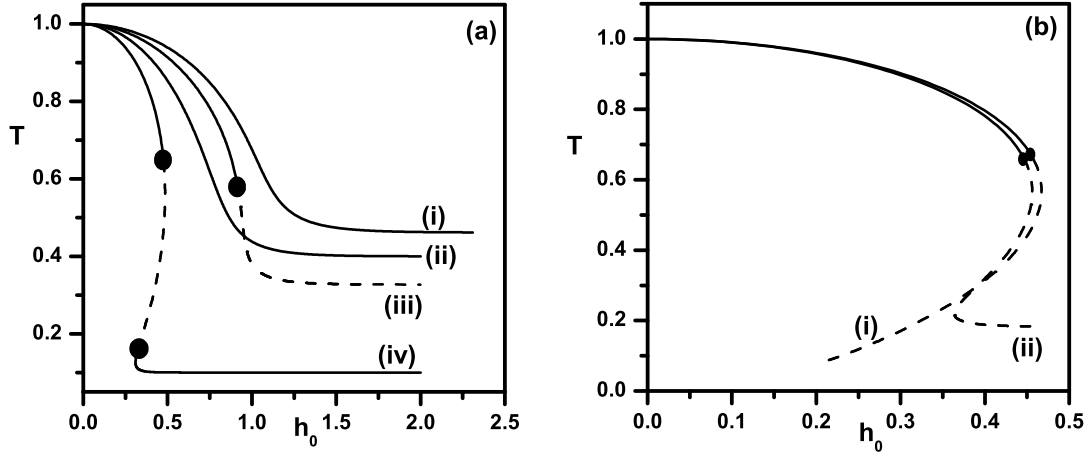


Fig. 6. Phase diagram of the Hamiltonian (4) in the $(h_0 - T)$ plane for specific values of p and q . The solid curve is a line of critical points and the dashed one is a line of first-order phase transitions, joined smoothly by a tricritical point (full circle). The system has, in general, second-order transition as in panel (a) for $p = 0.35, q = 0.0$, curve (i) and $p = 0.35, q = 0.20$, curve (ii); however, for $p = 0.05, q = 0.40$, curve (iii), it displays both phase transitions, FOPT and SOPT joined by a tricritical point, whereas for $p = 0.45, q = 0.45$, curve (iv), the system has two tricritical points, occurring also for other values of p and q . In the latter case, the system displays reentrance, as well. In panel (b) the phenomenon of reentrance is shown in enlargement; for $p = 0.45, q = 0.50$ (curve (i)) the system remains in the PM phase for low temperatures and medium random fields, whereas for $p = 0.45, q = 0.55$ (curve (ii)) it returns to the FM phase for low temperatures and higher random fields. Temperature T is expressed in units of (Jz/k) , i.e., $T \equiv kT/(Jz)$.

PM to *FM* with the system, now, in the *FM* phase for low temperatures and high fields, Fig. 6(b(ii)); occasionally, the region of the *FM* phase shrinks significantly, Fig. 6(a(iv)). The vanishing of magnetization for high values of p ($p \sim 0.45$, 45% up-spins) and small q values (where one would expect a nonzero magnetization) can also be due to the presence of the auxiliary field V_0 , which annihilates the excess magnetization making the system to behave like an antiferromagnet. This phenomenon is evident from the bending of the phase transition lines on lowering the temperature thus forming an inverted "C" ("boomerang" shape) and appearing in enlargement in Fig. 6(b). The respective p values lie in the interval $[0, 0.50]$, whereas the q values lie in a much smaller interval, namely, $[0.40, 0.55]$; the only exception is for $p = 0.45$ when q takes on values in the interval $[0, 0.55]$. However, within the MFA reentrance may lead to nonphysical values (negative) for the specific heat, since energy will also present reentrant behavior as magnetization because energy, within MFA, is proportional to the magnetization squared thus behaving similarly. The two successive transitions can be of first or second order depending on p, q and h_0 .

Solving Eq. (14) to determine the phase diagram, the magnetization is also calculated for either phase transition. The condition $A = 1$ or $\alpha F_2(\beta, V_0, h_0) = 1$ leads to

$$p t_+^2 + q t_-^2 + r t_0^2 = \frac{\alpha - 1}{\alpha} \quad (20)$$

and, by setting $T_2 \equiv p t_+^2 + q t_-^2 + r t_0^2$, Eq. (20) can be written as,

$$T_2 = \frac{\alpha - 1}{\alpha} \quad (21)$$

Inverting Eq. (21) the respective temperature for either phase transition can be determined,

$$\frac{kT}{Jz} = 1 - T_2 \quad (22)$$

In order to specify the type of the transition, the sign of $C \equiv \alpha^3 F_4/6$ is checked; however, to facilitate the calculations, the quantity C is rewritten as,

$$C = \frac{\alpha^3}{3} [4T_2 - 3T_4 - 1] = \alpha^3 [1 - T_4 - \frac{4}{3\alpha}] \quad (23)$$

using Eq. (21) and $T_4 = p t_+^4 + q t_-^4 + r t_0^4$. For an SOPT, C is negative [8,9,26], then Eq. (23) yields,

$$T_4 > 1 - \frac{4}{3\alpha} \quad (24)$$

otherwise if

$$T_4 < 1 - \frac{4}{3\alpha} \quad (25)$$

the resulting transition is an FOPT. However, in order to determine the magnetization for an FOPT the expression (13) is combined with the equality of the respective free energies,

$$F(M = 0) = F(M \neq 0) \quad (26)$$

or,

$$M^2 = F_2 \alpha M^2 + \frac{F_3}{3} \alpha^2 M^3 + \frac{F_4}{12} \alpha^3 M^4 + \frac{F_6}{360} \alpha^5 M^6 \quad (27)$$

Combining Eqs. (13), (27) and using the condition $\alpha F_2 = 1$, leads to,

$$F_6 \omega^3 + 10 F_4 \omega = 0 \quad (28)$$

which is split up into two equations, the first is $\omega_1 = 0$ or, equivalently, $M_1 = 0$, PM phase, whereas the other is,

$$F_6 \omega^2 + 10 F_4 = 0 \quad (29)$$

leading to the nonzero solutions, FM phase,

$$\omega_{2,3} = \pm \sqrt{-10F_4/F_6} \quad (30)$$

for which $F_6 < 0$ for stability requirements as far as $F_4 > 0$ for an FOPT; the value for F_3 , consistent with (13) and (27), is $F_3 = -(F_4/6)\sqrt{-10F_4/F_6}$ for the positive root in (30) and $F_3 = (F_4/6)\sqrt{-10F_4/F_6}$ for the respective negative root. From the solution of this equation we can extract the magnetizations $M_{2,3}$, since the temperature (kT/zJ) is already known from (22).

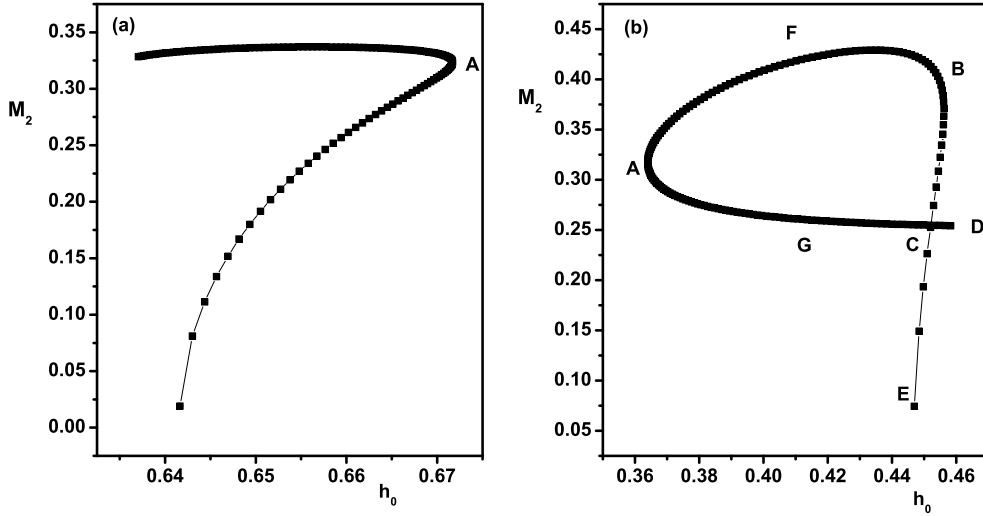


Fig. 7. Magnetization profile vs. h_0 , for (a) $p = 0.25, q = 0.55$, point A is a critical point. (b) $p = 0.45, q = 0.55$ with three critical points A, B, C. Point A is a normal critical point. B a critical-end-point; the two critical phases are on either branch at point B coexisting with the non critical phase on the branch CD. C a double critical-end-point; the first group of the two critical phases are on the branches CGA and CE coexisting with the non critical phase on the branch BF, the second group of two critical phases are on the branches CD and CB coexisting with the non critical phase on the branch BF. This plot represents a closed-loop phase diagram with left-hand and right-hand-side critical points.

Considering, now, the randomness strength h_0 as a control parameter, similarly as temperature, we study the variation of the non-zero positive magnetization $M_2 = \omega_2 * (kT/(Jz))$ for an FOPT, Eq. (30), coexisting in equilibrium with the zero magnetization $M_1 = 0$, by calculating the respective magnetization profile as a function of h_0 for specific values of p and q ; the negative $M_3 = -M_2$ behaves analogously. These profiles appear in Fig. 7 as functions of h_0 and look like the ones when the temperature is the control parameter for an FOPT. For $p = 0.25, q = 0.55$ the magnetization has a normal critical point Fig. 7(a), whereas for $p = 0.45, q = 0.55$ this structure is

transformed into a closed loop magnetization profile, closed miscibility gap, Fig. 7(b); it possesses multiple critical points, the point A is a pure critical point, whereas B is a critical-end-point (CEP) because the thermodynamic states along the branches BF and BC become identical at the point B in the presence of the third non critical phase (spectator phase) along the branch CD for the same value of h_0 ; the CEPs can be found in binary fluid mixtures, superfluids, binary alloys, liquid crystals, ferromagnets, ferroelectrics, etc. [26,44,45,46,47,48,49,50,51]. A simple example for the occurrence of the CEP is to consider a binary fluid mixture with the proper thermodynamic conditions so that three fluid phases can coexist in equilibrium and enclosed in a capsule; these phases are called V (vapor, top phase), L_1 (liquid phase rich in species 1, middle phase) and L_2 (liquid phase rich in species 2, bottom phase). The top and middle phases are separated by the interface (VL_1), as well as the middle and the bottom ones by the interface (L_1L_2); if the thermodynamic conditions are such that the interface VL_1 disappears with the phases V , L_1 becoming identical in the presence of the non critical L_2 -phase, then that thermodynamic state is a CEP at the temperature $T_{VL_1}^{CEP}$ and the new phase VL_1 coexists with the L_2 one in equilibrium. However, if we consider that the interface L_1L_2 disappears with the phases L_1 , L_2 becoming identical in the presence of the non critical V -phase, then that thermodynamic state is a CEP at the temperature $T_{L_1L_2}^{CEP}$ and the new phase L_1L_2 coexists with the V one in equilibrium. Another occurrence of a CEP is the double critical end point (DCEP) where two critical lines end simultaneously at a first order phase boundary; such a situation appears at the point C , where the two critical phases on the branches CGA and CE become identical at the point C in the presence of the non critical phase on the branch BF; the second group of two critical phases are on the branches CD and CB that become identical in the presence of the non critical phase on the branch BF; these are also observed in binary fluid mixtures, metamagnets, Ising antiferromagnets with next-nearest-neighbor interactions, RFIM, three dimensional antiferromagnetic spin-1 Blume-Capel model [26,45,51,52]. Point A can also be considered as a left-hand side critical point and B, C right-hand side ones.

Now, we consider the values of p and q for which the system exhibits only an SOPT; Eq. (14) takes the form for $A = 1$,

$$F_6\omega^5 + 20F_4\omega^3 + 60F_3\omega^2 = 0 \quad (31)$$

The value $\omega_1 = 0$ (two-fold) is again a solution or, equivalently, $M_1 = 0$ (PM phase); the other three roots are the solutions to the third degree equation,

$$F_6\omega^3 + 20F_4\omega + 60F_3 = 0 \quad (32)$$

which, depending on the values of p, q and h_0 , can have either only one real

non zero solution if $\Delta = q^3 + r^2 \geq 0$ ($r = -30F_3/F_6$, $q = (20F_4)/(3F_6)$),

$$\omega_2 = \sqrt[3]{r + \sqrt{\Delta}} + \sqrt[3]{r - \sqrt{\Delta}} \quad (33)$$

or three real non zero ones for $\Delta < 0$,

$$\begin{aligned} \omega_2 &= 2\sqrt[3]{\rho} \cos(\theta/3) \\ \omega_3 &= -\sqrt[3]{\rho} [\cos(\theta/3) + \sqrt{3}\sin(\theta/3)] \\ \omega_4 &= -\sqrt[3]{\rho} [\cos(\theta/3) - \sqrt{3}\sin(\theta/3)] \end{aligned} \quad (34)$$

where $\rho = \sqrt{r^2 - \Delta}$, $\theta = \arctan(\sqrt{-\Delta}/r)$ and $M_i = \omega_i * (kT/(Jz))$, $i = 2, 3, 4$. The solutions for an SOPT are classified into two groups, group-1 includes the zero solution ($M_1 = 0$) and the single nonzero one M_2 of (33), whereas group-2 includes again the zero solution and the nonzero ones M_2, M_3, M_4 of (34). Depending on the value of p, q and h_0 , there can be transitions between these two groups. The majority of the SOPT solutions belong to the group-2; for given values of p and q the stable solution is the zero (PM phase) for small values of h_0 , whereas for higher ones the most probable to be stable is the M_2 solution. For the group-1, the zero solution is stable for small values of h_0 and the M_2 solution for higher values.

The investigation was also extended towards the zero-temperature case, $T = 0$; in this case the free energy (7) reduces to,

$$\begin{aligned} \frac{1}{N} \langle F \rangle_h &= \frac{1}{2} zJM^2 - \frac{1}{\beta} \langle \ln \{ 2 \cosh[\beta(zJM + h_i)] \} \rangle_h \\ &= \frac{1}{2} zJM^2 - \langle |zJM + h| \rangle_h \\ &= \frac{1}{2} zJM^2 - p|zJM + h_0| - q|zJM - h_0| - r|zJM| \end{aligned} \quad (35)$$

the external potential was omitted. Applying the equilibrium condition $dF/dM = 0$ to (35) we get,

$$M = p \frac{|zJM + h_0|}{zJM + h_0} + q \frac{|zJM - h_0|}{zJM - h_0} + r \frac{|zJM|}{zJM} \quad (36)$$

Analyzing Eq. (36) we find a variety of solutions because of the greater number of degrees of freedom than in the case of the bimodal PDF [26]. The solution $M = 1$ is a stable one for $p + r > h_0/zJ$, whereas for $p + r < h_0/zJ$ the stable one is $M = 1 - 2q$. If we consider the symmetric trimodal PDF ($p = q = \frac{1-r}{2}$), the results found by Sebastianes and Saxena [28] are recovered, that is, the

former result ($M = 1$) is stable for $\frac{1-r}{2} > h_0/zJ$, whereas the latter ($M = r$) for $\frac{1-r}{2} < h_0/zJ$, using the current notation. Across the line $h_0/zJ = p + r$ in the (p, h_0) -plane a first-order phase transition occurs between the two ordered phases with $M = 1$ and $M = 1 - 2q$. In addition to the aforementioned two solutions, there are more ones; the result $M = 2(p + q) - 1$ is stable for $p - r > h_0/zJ$, $M = 2p - 1$ for $p - r < h_0/zJ$, $M = 1 - 2p$ for $r - p > h_0/zJ$, $M = 1 - 2(p + q)$ for $r - p < h_0/zJ$ and $M = -1$ for $p + r + h_0 > 0/zJ$. In the first case, $M = 1$, the condition $p > (h_0/(zJ))$ implies that the exchange interaction J is much stronger than randomness h_0 , their ratio is always smaller than one, thus forcing the system's spins to order according to the first term in (4). The alternative condition $p < (h_0/(zJ))$ implies, now, that randomness is not any more negligible but strong enough to influence significantly the spins enforcing a p -fraction of them to point up and a q -fraction down, to randomly align with the local fields, thus, practically, it dominates, so to speak, over the first term in Eq. (4) so that $M = p - q + r = 1 - 2q$.

4 Conclusions and discussions

In the current treatment we have determined the phase diagram and discussed some critical phenomena of the Ising model under the influence of a trimodal random field, an extension of the bimodal one, to allow for the existence of non magnetic particles or vacancies in the system, for arbitrary values of the probabilities p and q via the mean-field approximation. The competition between the ordering effects and the randomness induces a rich phase diagram. The system is strongly influenced by the random field, which establishes a new competition favoring disorder; this is obvious by the appearance of first-order transitions and tricritical points, in addition to the second-order transitions, for some values of p and q ; the tricritical point temperature has various modes of variation as a function of p and q . The trimodal distribution induces reentrant behavior for the appropriate range of p, q and random field h_0 . For some values of p and q the system can be found either in the PM phase or in the FM phase for low temperatures and medium and/or high random fields; occasionally, the part of the phase diagram allocated to the FM phase shrinks significantly. A direct consequence of the asymmetric PDF ($p \neq q$) is the existence of residual mean magnetization, a result of $\langle h_i \rangle_h = (p - q)h_0$, making the TCP non zero magnetization M_2 to be the stable one in comparison to the zero one, M_1 ; however, for $p \neq q$ (symmetric PDF) the residual mean magnetization vanishes as well as the initially TCP non zero magnetization so that $M_2 \equiv M_1$, which is now the stable one. Both asymmetric PDFs, bimodal and trimodal, confirm the existence of a TCP and, nevertheless, yield similar magnetization profiles as well as reentrance; however, the trimodal one predicts also the existence of a second TCP. The tricritical point temperatures for the bimodal and trimodal PDF's are independent of the random field strength h_0 ; they depend only on the probability p and p, q , respectively.

Griffiths extended the notion of the critical point to the so-called multicritical points, e.g., tricritical, critical-end-point, double critical-end-point, fourth-order, ordered critical point, etc. [53]; however, in order to describe these points (except the first two) the considered expansion of the free energy (10) has to be extended to higher-order terms [27,37,54,55] so that the stability criteria for such a point are satisfied, but this is beyond the scope of the current research.

The Landau theory breaks down close to the critical point (non-classical region) because as the transition temperature is approached the fluctuations become important and non-classical behavior is observed. A relative criterion, called Ginzburg criterion, determines how close to the transition temperature the true critical behavior is revealed, or, in other words, it governs the validity of the Landau theory close to a critical point [56]. This criterion relies on any thermodynamic quantity but the specific heat is usually considered for determining the critical region around T_c where the mean field solution cannot describe correctly the phase transition. The Landau theory is valid for lattice dimensionality greater than or equal to the upper critical dimension $d_u = 4$ in case of presence only of thermal fluctuations. However, in the current case the presence of random fields enhances fluctuations causing the critical region to be wider than the one due only to the thermal fluctuations [57,58] and the upper critical dimension is increased by 2 to $d_u = 6$.

Our results indicate that on increasing the complexity of the model system new phenomena can be revealed as in the current case of including asymmetry in the PDF; this inclusion induces drastic changes on the phase diagram, such as reentrance and two TCPs, thus confirming the necessity of treating the partial probabilities (p, q, r) of the PDF in the most general aspect to get the complete phase diagram. A similar situation appears in the model systems in Refs. [23,24] wherein the considered complexity has revealed a rich variety of phase diagrams with known and new multicritical points. The results obtained in the current investigation by using the MFA need further analysis; these can provide a basis for a comprehensive analysis by more sophisticated methods as well as experimental implementation. However, they are of no less importance, since they show, nevertheless, the expected phenomena to be observed.

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